Recall Fourier Series

PHYS 331
Oct. 30, 2023

- Can represent any periodic fcn as an infinite series of trig. fcns.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

$$\omega = \frac{2\pi}{T}$$ T is period of f(t)

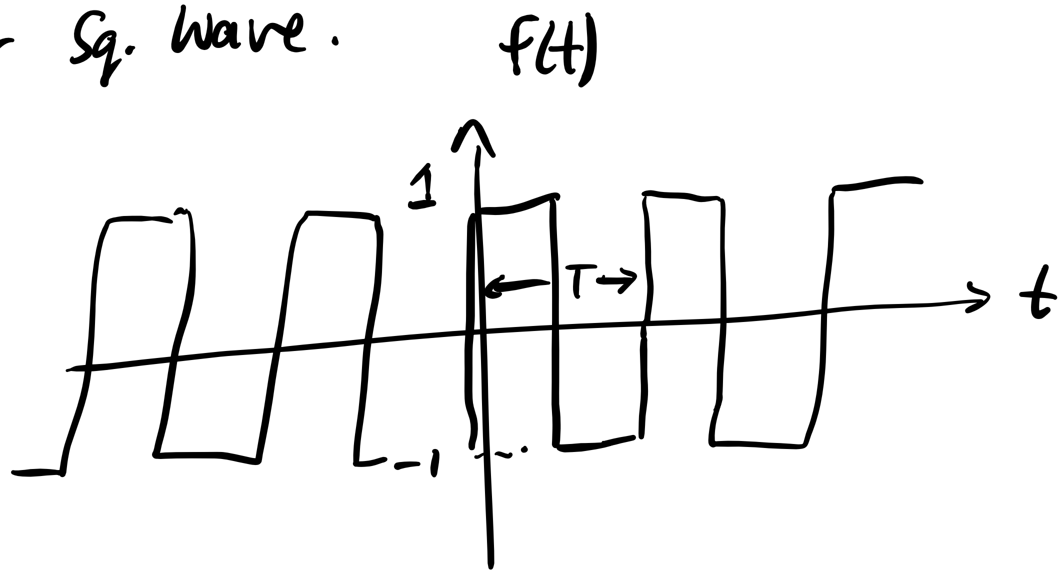
The coefficients are given by:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

For Sq. wave.



$a_n = 0 \quad \forall$

$b_n = 0 \quad \forall \; n$ even

$b_1 = \frac{4}{\pi} \qquad b_3 = \frac{4}{3\pi} \qquad b_5 = \frac{4}{5\pi} \quad \ldots$

$$f(t) = \frac{4}{\pi}\left[\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right]$$

Postulate that we can also express Fourier series in terms of complex exponentials.

$$\left(e^{\pm j n\omega t} = \cos n\omega t \pm j \sin n\omega t\right)$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

If this is true, need to find c_n coefficients.

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \underbrace{\sum_{n=-\infty}^{-1} c_n e^{jn\omega t}}_{n \to -n}$$

$$\sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t}$$

$$= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{jn\omega t} + c_{-n} e^{-jn\omega t} \right)$$

use Euler's Eq'n.

$$f(t) = c_0 + \sum_{n=1}^{\infty} \left[c_n (\cos n\omega t + j \sin n\omega t) + c_{-n} (\cos n\omega t - j \sin n\omega t) \right]$$

$$= c_0 + \sum_{n=1}^{\infty} \left[(c_n + c_{-n}) \cos n\omega t + j(c_n - c_{-n}) \sin n\omega t \right]$$

In the same form as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos n\omega t + b_n \sin n\omega t\right]$$

$$c_0 = \frac{a_0}{2} \qquad c_n + c_{-n} = a_n \qquad j(c_n - c_{-n}) = b_n$$

$$\Rightarrow \quad c_n = \frac{a_n - j b_n}{2}$$

$$c_{-n} = \frac{a_n + j b_n}{2}$$

$$\therefore c_n = \frac{\cancel{2}}{T}\int_{-T/2}^{T/2} f(t)\,\frac{\overbrace{\left[\cos n\omega t - j \sin n\omega t\right]}^{e^{-jn\omega t}}}{\cancel{2}}\,dt$$

$$\boxed{\therefore c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega t}\,dt}$$

Valid $\forall$ n, including $n=0$.

For square wave, the exponential form of the Fourier series is:

$$C_n = \frac{\cancel{a_n}^{\,0} - jb_n}{2} = \frac{b_n}{2j}$$

$$C_{-n} = \frac{\cancel{a_n}^{\,0} + jb_n}{2} = \frac{jb_n}{2} = -\frac{b_n}{2j} = -C_n$$

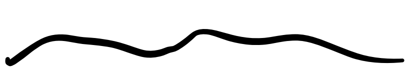
$$C_{\pm 1} = \pm\frac{4}{\pi}\frac{1}{2j} = \pm\frac{2}{j\pi}$$

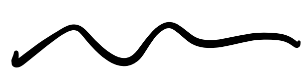
$$C_{\pm 3} = \pm\frac{4}{\pi}\frac{1}{3}\frac{1}{2j} = \pm\frac{2}{3j\pi}$$

$$\vdots$$

$$f(t) = \frac{2}{j\pi}\left[\cdots - \frac{1}{5}e^{-j5\omega t} - \frac{1}{3}e^{-j3\omega t} - e^{-j\omega t} + e^{j\omega t} + \frac{1}{3}e^{j3\omega t} + \frac{1}{5}e^{j5\omega t} + \cdots\right]$$

$2j\sin 5\omega t$ $\qquad$ $2j\sin 3\omega t$





$$= \frac{2}{j\pi}\left[\cdots + \frac{1}{5}\left(e^{j5\omega t} - e^{-j5\omega t}\right) + \frac{1}{3}\left(e^{j3\omega t} - e^{-j3\omega t}\right) + \underbrace{\left(e^{j\omega t} - e^{-j\omega t}\right)}_{2j\sin\omega t}\right]$$

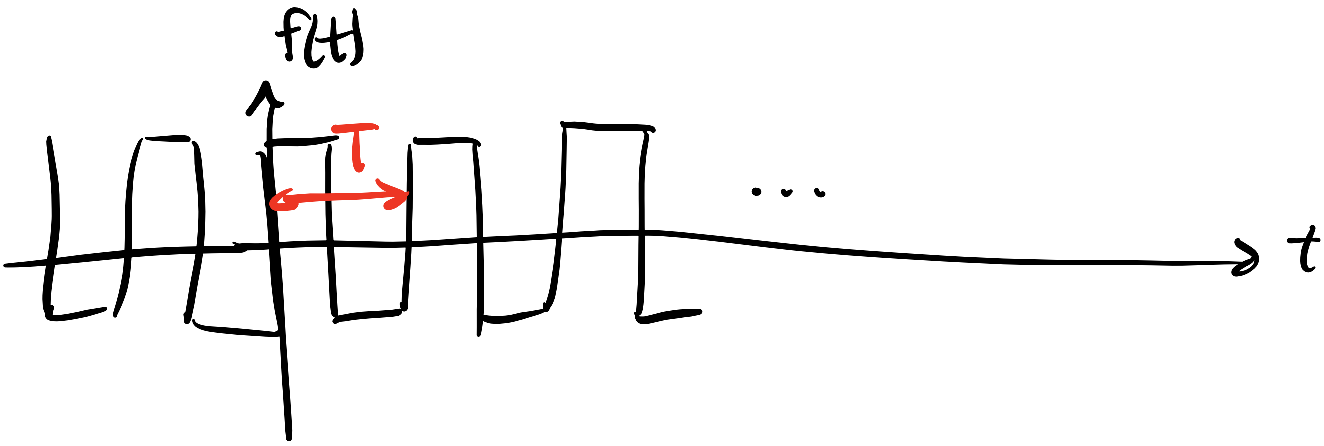
$$= \frac{4}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right]$$

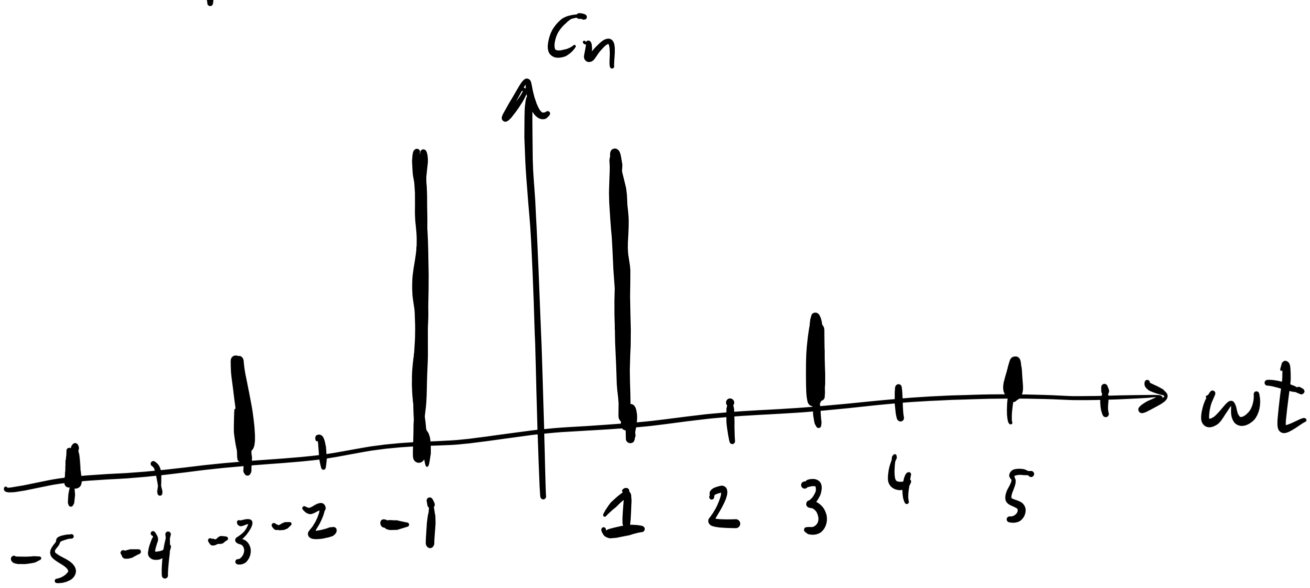
same as before!

<u>Summary</u>: Fourier series for periodic fcns in terms of complex exponentials.

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega t}\, dt}$$

For Sq. wave:





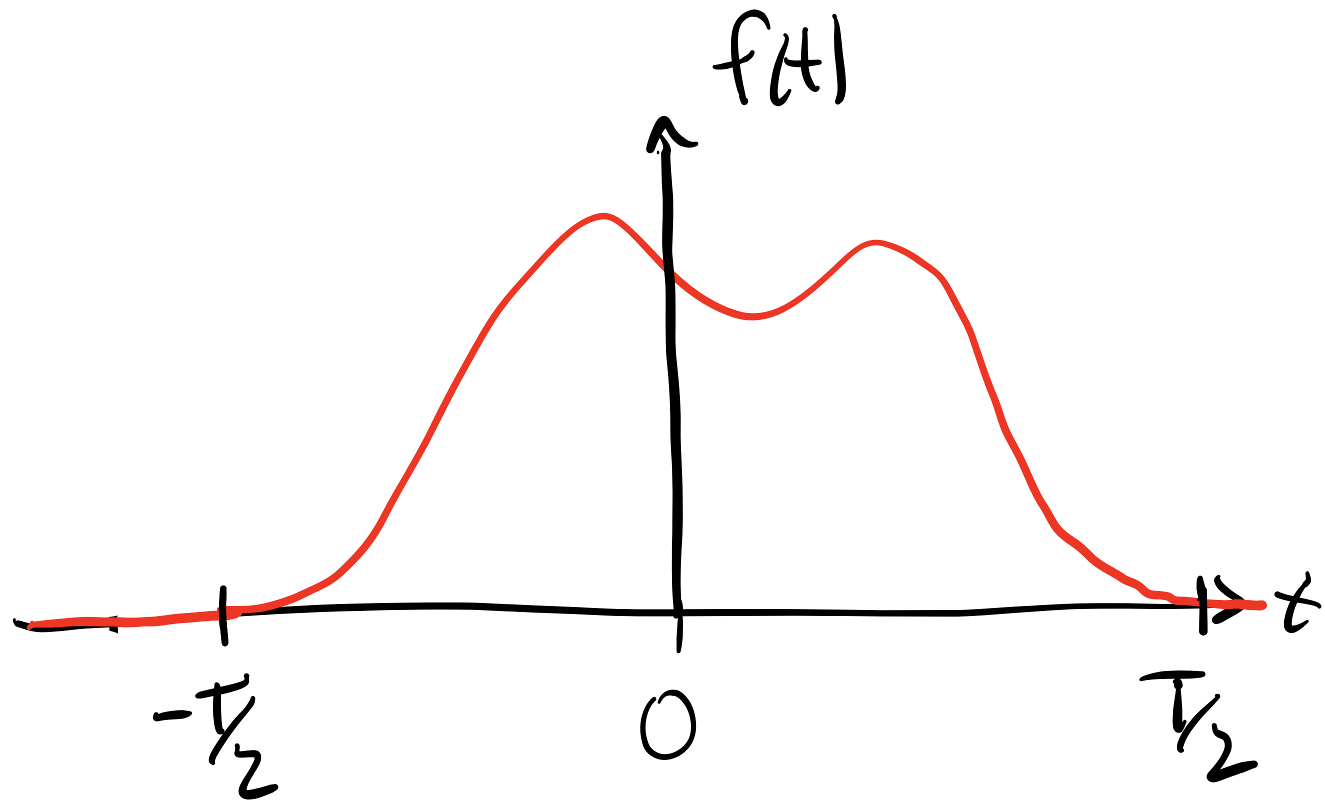
nonzero freq.: $\frac{2\pi}{T}$, $\frac{2\pi\cdot 3}{T}$, $\frac{2\pi\cdot 5}{T}$, ...

Observations: - Periodic fcns are made up of discrete freq. components.

- Spacing between nonzero freq. components decreases as T increases. In limit that $T \to \infty$, expect continuous range of nonzero freq. components.

Want to generalize the Fourier series so that we can examine freq. content of some signal it time (pulse) $f(t)$ that is <u>not</u> periodic.

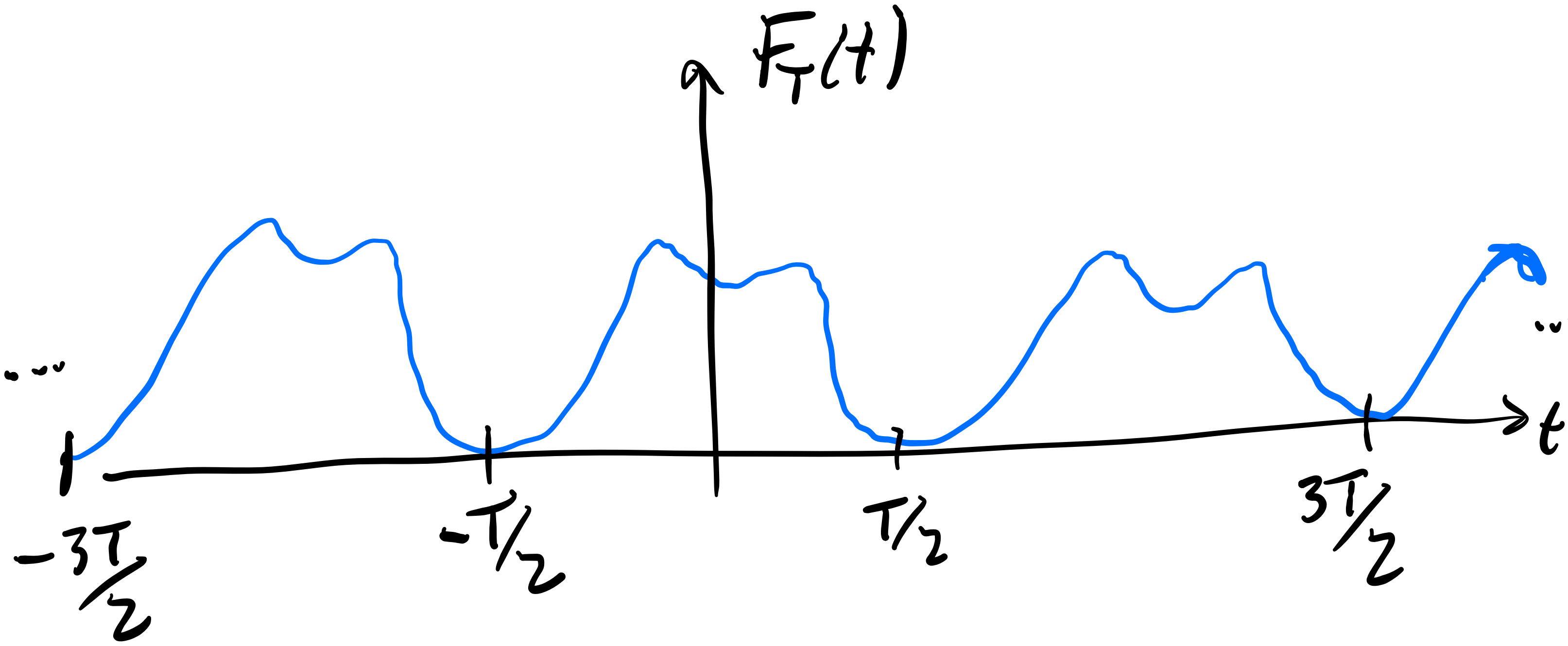
Eg. $f(t) = 0 \quad \forall \quad |t| \geq \frac{T}{2}$



Now, let's construct a periodic fcn $F_T(t)$ using $f(t)$.

Require $F_T(t) = f(t)$ on $-\frac{T}{2} < t < \frac{T}{2}$

and $F_T(t)$ is periodic w/ period T.



Then, for $-T/2 < t < T/2$

$$f(t) = F_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t} \quad \left(\omega = \frac{2\pi}{T}\right)$$

can be expressed as Fourier series

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} F_T(t)\, e^{-jn\frac{2\pi}{T}t}$$

Step 1: Replace $F_T(t)$ w/ $f(t)$ in c_n integral (valid b/c integral on $-T/2 < t < T/2$)

$$\therefore c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\frac{2\pi}{T}t}\, dt$$

Step 2: Change limits of integration from $-T/2$ to $T/2$ to $-\infty$ to ∞.
(valid since $f(t) = 0$ outside $-T/2 < t < T/2$).

$$c_n = \frac{1}{T}\int_{-\infty}^{\infty} f(t)\, e^{-j\underbrace{n\frac{2\pi}{T}}_{\omega_n}t}\, dt$$

n^{th} freq. component $\omega_n = \frac{2\pi n}{T}$

The spacing between adjacent freq. components is

$$\omega_{n+1} - \omega_n \equiv \Delta\omega = \frac{2\pi}{T}$$

∴ can express $\frac{1}{T}$ in front of integral as $\frac{\Delta\omega}{2\pi}$.

$$c_n = \underbrace{\frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega_n t}\, dt}$$

$$\equiv \hat{f}(\omega_n) \quad !$$

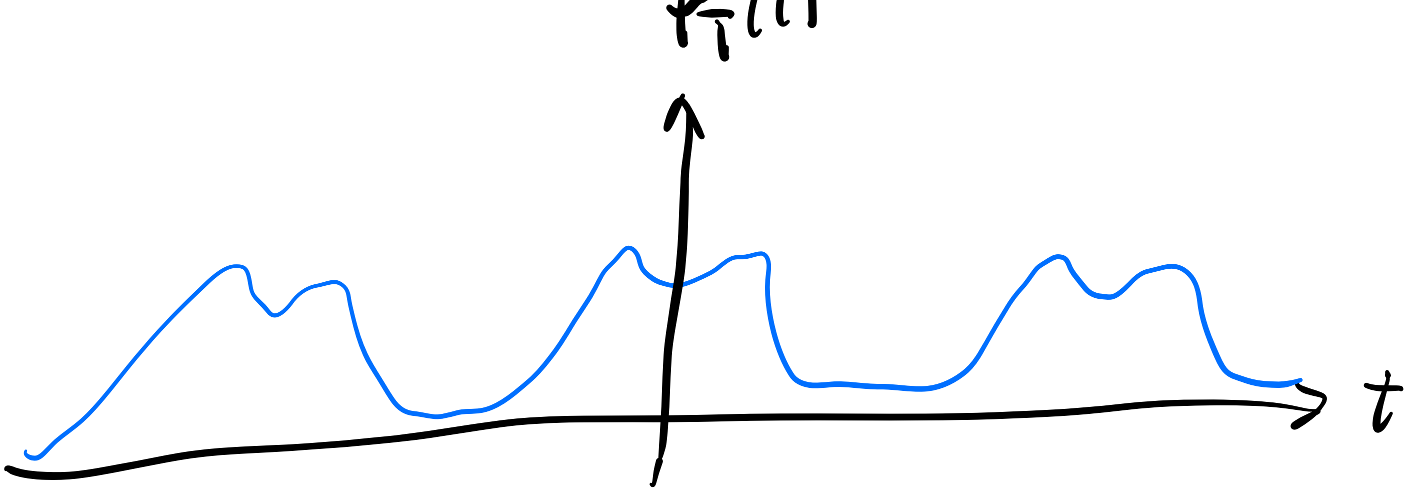
$$\therefore \; c_n = \frac{1}{2\pi}\hat{f}(\omega_n)\Delta\omega$$

Return to ⊛:

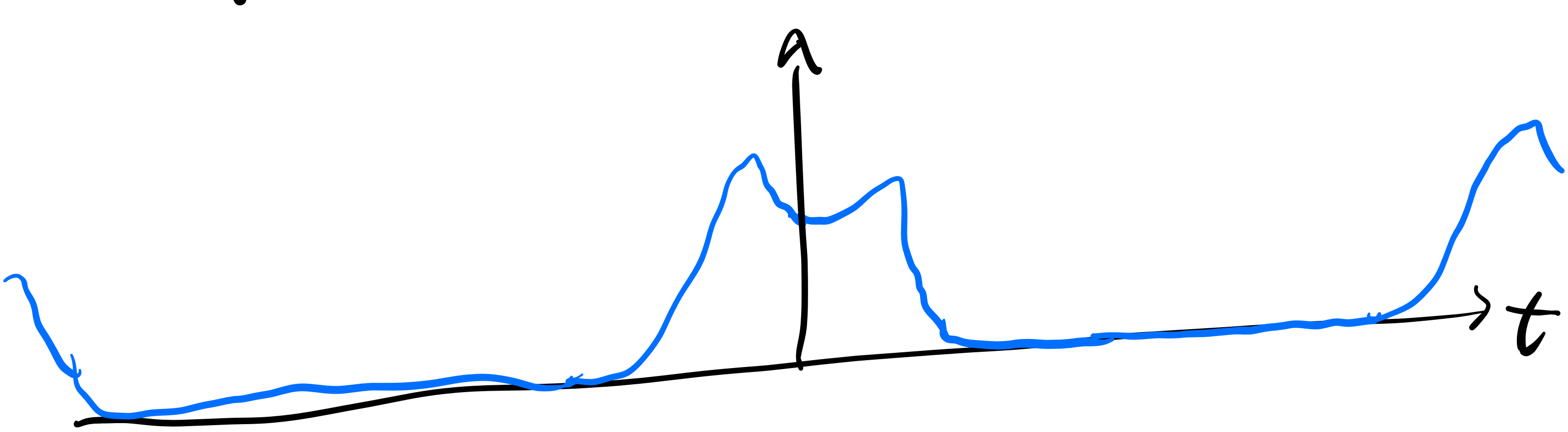
On $-T/2 < t < T/2$ we have

$$\# \quad f(t) = F_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{F}(\omega_n)\, e^{j\omega_n t}\,\Delta\omega$$

small T



large T



In limit $T \to \infty$, # becomes:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{j\omega t}\, d\omega}$$

Inverse Fourier transform of $\hat{f}(\omega)$.

Takes a fcn of freq. $\hat{f}(w)$ and transforms it to a fcn of time $f(t)$.

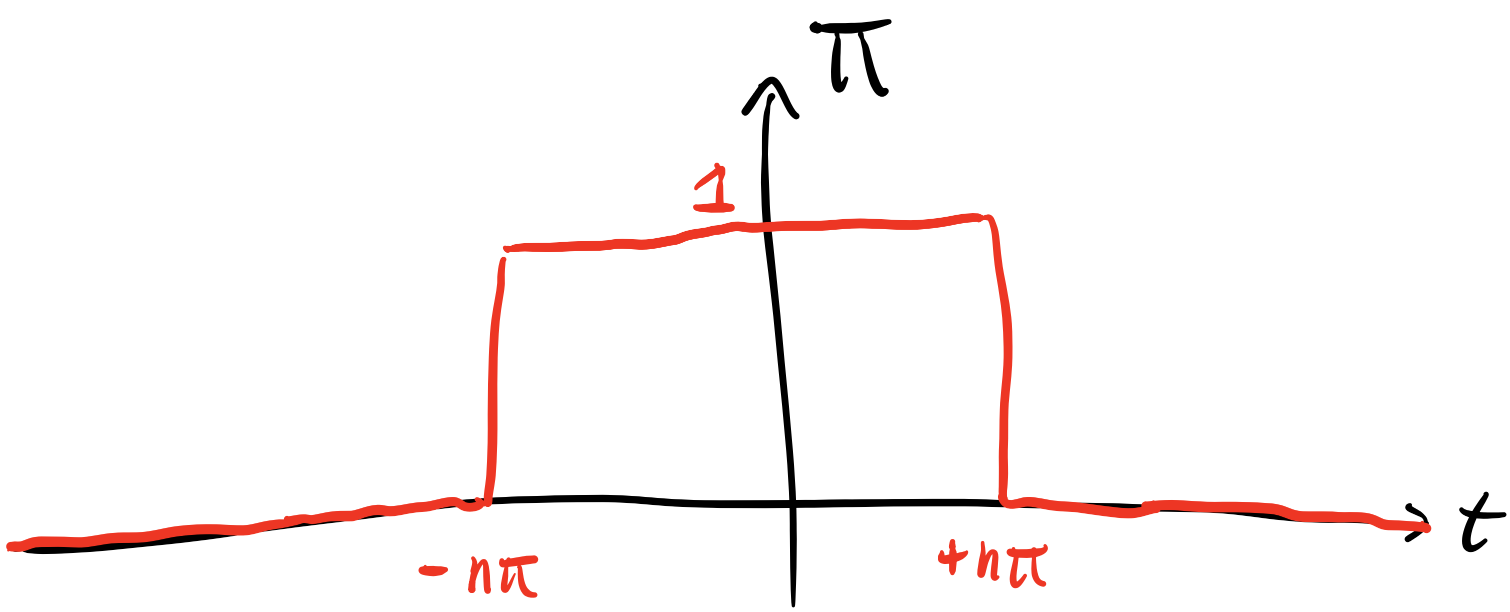
(!) $$\boxed{\hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{-jwt} \, dt}$$

Fourier transform of $f(t)$.

Takes a fcn of time $f(t)$ and transforms it to fcn of freq. $\hat{f}(w)$.

Fourier Trasform Example.

Box fcn Π



$$\Pi(t) = \begin{cases} 1 & -n\pi < t < n\pi \\ 1/2 & t = \pm n\pi \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{\Pi}(\omega) = \int_{-\infty}^{\infty} \Pi(t) e^{-j\omega t} dt$$

$$= \int_{-n\pi}^{n\pi} 1 \cdot e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-n\pi}^{n\pi}$$

$$= -\frac{1}{j\omega} \left[e^{-jn\omega\pi} - e^{jn\omega\pi} \right]$$

$$= \frac{2}{\omega} \left[\frac{e^{jn\omega\pi} - e^{-jn\omega\pi}}{2j} \right]$$

$$\sin(n\pi w)$$

$$\therefore \quad \hat{\Pi}(w) = 2 \underbrace{\frac{\sin n\pi w}{n\pi w}}_{\mathrm{sinc}(wn)} n\pi$$

$$\boxed{\therefore \quad \hat{\Pi}(w) = 2n\pi \, \mathrm{sinc}(nw)}$$